1/23/26 Collect HW

Thm A lin syst w/ n equs in n vars has a unique sol'n iff the rank of the coeff matrix is n. In this case, if B is the augmented matrix then

$$RREF(B) = \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & & 1 \end{matrix}}_{\substack{n \times n \\ \text{identity} \\ \text{matrix}}} & \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \end{array}\right]$$

pf

sol'n is

$$x_1 = b_1$$
$$x_2 = b_2$$
$$\vdots$$
$$x_n = b_n$$

//

Ex $\mathcal{L}$ = linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 7x_3 &= 1 \\ 3x_1 + 8x_2 + 12x_3 &= 1 \end{aligned}$$

Augmented matrix: (I'll skip the steps in class.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & 1 \\ 3 & 8 & 12 & 1 \end{array}\right] \to \boxed{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]} \leftarrow \text{skip}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$x_1 = 3$$
$$x_2 = -1$$
$$x_3 = 0$$

(check equations to confirm -

$3 + 2(-1) + 3(0) = 1$

$2(3) + 5(-1) + 7(0) = 1$

$3(3) + 8(-1) + 12(0) = 1$)

Matrix algebra

Recall A is an $n \times m$ matrix if it has n rows & m columns.

We say the size of A is $n \times m$.

- We can add two matrices as long as they're the same size (see Def 1.3.5).
  - the result also has the same size
- Scalar multiple:

  Given a scalar k and an $n \times m$ matrix A we form a new $n \times m$ matrix kA. (Def 1.3.5)

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

$k = 10$

$$A + B = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 6 & 10 \end{bmatrix}$$

$$kA = \begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{bmatrix}$$

(Matrix mult is a bit more complicated. We'll come to that shortly.)

Def Given vectors $\vec{v}, \vec{w}$ of the same size, we define their dot product as follows.

Say $\vec{v} = [v_1, \ldots, v_n]$

$\vec{w} = [w_1, \ldots, w_n]$.

Then $\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n$.

Note the dot product of 2 vectors is a number, not a vector.

The vectors have to be in the same vector space $\mathbb{R}^n$ (ie same # of entries)

Ex $n = 3$ $\vec{v} = [1 \;\; 2 \;\; 3]$

$\vec{w} = [4 \;\; 5 \;\; 1]$

$\vec{v} \cdot \vec{w} = 4 + 10 + 3 = 17$

<u>Remark</u> We'll define matrix mult shortly, + this is a special case

<u>Ex</u> in last example

1st example of matrix mult.

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = 4 + 10 + 3 = 17$$

dot product

$(1 \times 3) \cdot (3 \times 1)$

But first an intermediate situation

<u>Def</u> Let A be an $n \times m$ matrix and $\vec{x}$ a vector in $\mathbb{R}^m$. Then

$A\vec{x}$ is as defined in Def 1.3.7. in the book.

Just do an example

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ $\quad 3 \times 4$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \in \mathbb{R}^4$

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 + 2 + 6 - 8 \\ 0 + 3 - 2 - 4 \\ 0 + 1 + 4 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$$

$(3 \times 4) \quad (4 \times 1) = (3 \times 1)$

in terms of dot products

$$= \begin{bmatrix} (1^{st} \text{ row}) \cdot \vec{x} \\ (2^{nd} \text{ row}) \cdot \vec{x} \\ (3^{rd} \text{ row}) \cdot x \end{bmatrix}$$

Ex

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3×3 identity matrix

But

$$\underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

is <u>undefined</u>

Ex the lin syst

$$\left| \begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 8 \\ 4x_1 - x_2 + 4x_3 &= 10 \end{aligned} \right|$$

is the same as the matrix product

$$\begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

It can also be written as

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ -1 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ 4 \end{bmatrix} x_3 = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

Ex Above we showed

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$$

This is also equal to

$$\begin{bmatrix}1\\2\\1\end{bmatrix}(0) + \begin{bmatrix}2\\3\\1\end{bmatrix}(1) + \begin{bmatrix}3\\-1\\2\end{bmatrix}(2) + \begin{bmatrix}4\\2\\3\end{bmatrix}(-2)$$

this is an example of a linear combination of the vectors

$$\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\2\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix}.$$

Bring some things together:

Ex (*) Is the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a linear comb. of the vectors $\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$?

We'll show that answering (*) is the same as solving a certain linear system!

(*) asks: can we find x_1, x_2, x_3 st

$$\begin{bmatrix}1\\1\\3\end{bmatrix}(x_1) + \begin{bmatrix}1\\0\\2\end{bmatrix}(x_2) + \begin{bmatrix}2\\1\\1\end{bmatrix}(x_3) = \begin{bmatrix}1\\2\\3\end{bmatrix}$$